Optical Fibers with Coupled Dispersive Modes

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Abstract—**Marcuse's coupled-power theory incorporates the effect of different modal group velocities as the only mechanism responsible for pulse broadening in multimode, imperfect, optical fibers. In this paper the earlier theory is generalized to include the bandwidth-dependent effects: material dispersion and the waveguide dispersion that occurs within each mode. It is predicted that the latter two effects may partially cancel one another, the degree of cancellation depending on the shape of the fiber's refractive-index profile. Whether this effect causes a significant reduction in the calculated pulsewidth is shown to depend on the amount of mode coupling.**

I. INTRODUCTION

CONSIDERATION of waveguide dispersion in multimode fibers has previously been given only to the case of uncoupled modes [1]. For the entire class of refractive-index profiles given by (33), it has been shown that there is no profile shape for which waveguide dispersion is significant.

Previous analyses of fibers with mode coupling caused by small, random, physical imperfections have shown that the optical power propagates at an average of the individual modal group velocities, and that the corresponding multimode time dispersion may be less than for a physically perfect fiber [2]. This desirable aspect of mode coupling is characterized by an "improvement factor" [3] that measures the ratio of the pulsewidth in the distorted fiber to that in a perfect one, assuming highly monochromatic excitation.

In this paper it is assumed that the excitation is wide band, and the earlier theory is generalized to include bandwidth-dependent effects: the waveguide dispersion that occurs within each mode and the bulk material dispersion. It is shown that the latter two effects determine a single dispersion parameter, the effect of which adds to that of multimode dispersion in rms fashion. This parameter is formed as the difference of the bulk material dispersion index and an average of the modal waveguide dispersion indices, so that the bandwidth-dependent dispersions partially cancel one another. Whether this effect causes a significant reduction in the calculated pulsewidth is shown to depend on the amount of mode coupling.

II. BACKGROUND

The propagation of temporally incoherent light in a physically imperfect multimode waveguide has been studied by means of coupled-mode theory. The modal fields $\mathcal{E}_\nu(r) \exp[j(\omega t - \beta_\nu z)]$ are the simplest time-harmonic solutions of the sourceless Maxwell's equations that satisfy the boundary conditions imposed by the waveguide. The most general time-harmonic field $E(r,z;\omega) \cdot \exp(j\omega t)$ guided by the fiber can be resolved uniquely as a linear superposition of the mode fields

$$E(r,z;\omega) = \sum_\nu I_\nu(z;\omega)\mathcal{E}_\nu(\mathbf{r}) \tag{1}$$

where factors $\exp(j\omega t)$ are suppressed. In the absence of physical imperfections, $I_\nu \exp(j\beta_\nu z)$ is independent of the axial coordinate z.

More generally, the fiber may have a small physical imperfection; the mode amplitudes may then be shown to satisfy a system of equations of the following form [3]:

$$\frac{dI_\nu}{dz} = -j\beta_\mu I_\nu + \sum_{\mu\neq\nu} K_{\nu\mu} f(z) I_\mu \tag{2}$$

where the function $f(z)$ is the actual physical imperfection. When the imperfections are numerous and irregular in occurrence, $f(z)$ is taken as a random process. For example, $f(z)$ might represent a random deviation of the fiber's radius from some average value.

Several successful treatments of this problem, both formal [3] and exact [4], [5] have recently appeared. Rather than attempting to solve (2) directly, each of these treatments has as its initial objective the reformulation of the random equations (2) as a set of deterministic equations on the second-order statistics of the mode amplitudes. For example, defining mode powers P_ν

$$P_\nu(z;\omega) \equiv \langle | I_\nu(z;\omega) |^2\rangle. \tag{3}$$

Marcuse has used (2) to obtain a set of coupled-power equations [3]

$$\frac{dP_\nu}{dz} = \sum_\mu h_{\nu\mu}(P_\mu - P_\nu) \tag{4}$$

that describe the evolution of the mode power distribution P_ν, for continuous-wave excitation. A description of pulse propagation was obtained by heuristically modifying (4) [3]

$$\frac{\partial P_\nu}{\partial z} + \frac{1}{v_\nu}\frac{\partial P_\nu}{\partial t} = \sum_\mu h_{\nu\mu}(P_\mu - P_\nu) \tag{5}$$

where v_ν is the group velocity of the νth mode.

Implicit in (5) is the assumption that the mode dispersion characteristics can be adequately modeled as straight lines [6], [7]; in the limit $h_{\nu\mu} \to 0$ (no mode coupling), (5) reduces to a description of propagation in nondispersive guides. Thus (5) may not be appropriate for wide-band excitation.

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III. MODIFICATION OF THE COUPLED-POWER EQUATIONS

The derivation of coupled-power equations given in [3] can be carried over step by step to derive an analogous set of equations for the covariances of the mode transfer functions [4], [6], [7]

$$I_{\nu\nu}(z;\omega,\omega_1) \equiv \langle I_\nu(z;\omega+\omega_1) I_\nu^*(z;\omega)\rangle. \quad (6)$$

When this was done, the following perturbation equation was obtained, ignoring the frequency dependence of the coupling [8], [7]:

$$\frac{dI_{\nu\nu}}{dz} + j\sigma_\nu I_{\nu\nu} = \sum_\mu h_{\nu\mu}(I_{\mu\mu} - I_{\nu\nu}) \quad (7)$$

defining $\sigma_\nu(\omega,\omega_1)$ by

$$\beta_\nu(\omega+\omega_1) = \beta_\nu(\omega) + \sigma_\nu(\omega,\omega_1) \quad (8)$$

where $\beta_\nu(\omega)$ is the propagation constant of the νth fiber mode in the absence of coupling.

The solution of (7) for $I_{\nu\nu}$ is related to the average mode power P_ν by

$$P_\nu(z;t) = 2\,\mathrm{Re}\left\{\int_0^\infty \frac{d\omega_1}{2\pi}\exp(j\omega_1 t)\left[\int_{-\infty}^\infty \frac{d\omega}{2\pi} I_{\nu\nu}(z;\omega_0+\omega,\omega_1)\right]\right\} \quad (9)$$

where ω_0 is the optical center frequency.

The dispersion relation of each perfect fiber mode is now given a quadratic approximation

$$\beta_\nu(\omega_0+\omega) \doteq \beta_{\nu 0} + \beta_{\nu 1}\omega + \beta_{\nu 2}\omega^2, \qquad |\omega| \ll \omega_0. \quad (10)$$

The coefficients $\beta_{\nu 0}$, $\beta_{\nu 1}$, and $\beta_{\nu 2}$ are functions of ω_0; they depend on both the material properties of the bulk fiber constituents and the fiber geometry [9]. Alternatively, one may write (10) as

$$\beta_\nu(\omega_0+\omega) \doteq \frac{\omega_0}{c}\left\{n_\nu + N_\nu\left(\frac{\omega}{\omega_0}\right) + \tfrac{1}{2}\eta_\nu\left(\frac{\omega}{\omega_0}\right)^2\right\} \quad (11)$$

where n_ν, N_ν, and η_ν are referred to, respectively, as the phase, group, and dispersion indices of the perfect fiber modes [10]; they may be related to the coefficients of (10) by equating like powers of ω.

From (8) and (10)

$$\sigma_\nu(\omega_0+\omega,\omega_1) \doteq \omega_1\beta_{\nu 1} + (2\omega\omega_1 + \omega_1^2)\beta_{\nu 2}. \quad (12)$$

With the trial solution

$$I_{\nu\nu}(z;\omega_0+\omega,\omega_1) = cB_\nu \exp(-\alpha z) \quad (13)$$

it follows from (7) and (12) that:

$$\{(-\alpha + j\beta_{\nu 1}\omega_1) + j\beta_{\nu 2}(2\omega\omega_1+\omega_1^2)\}B_\nu = \sum_\mu h_{\nu\mu}(B_\mu - B_\nu). \quad (14)$$

The second-order perturbation solution of (14) with $\beta_{\nu 2} = 0$ is given in [3], corresponding to the case of dispersionless modes. Assuming that the fiber is long enough to establish the steady-state mode power distribution, the solution of (14) with $\beta_{\nu 2} = 0$ may be written [3]

$$I_{\nu\nu}(z;\omega_0+\omega,\omega_1) = c(\omega,\omega_1)B_\nu \exp(-\alpha_M z) \quad (15)$$

$c(\omega,\omega_1)$ is determined by the excitation; B_ν is the steady-state mode power distribution normalized such that $\sum_\nu B_\nu^2 = 1$, and α_M is a complex propagation constant

$$\alpha_M(\omega_1) = \alpha_0 + j(\omega_1/v_0) + \omega_1^2\alpha_2. \quad (16)$$

It is now assumed that the effect of nonzero $\beta_{\nu 2}$ on the solution for $I_{\nu\nu}$ may be taken into account by a first-order perturbation correction to α_M

$$\alpha \doteq \alpha_M + j(2\omega\omega_1 + \omega_1^2)\bar\beta_2 \quad (17)$$

where

$$\bar\beta_2 = \sum_\nu B_\nu^2\beta_{\nu 2} = \bar\eta/2\omega_0 c \quad (18)$$

defines both $\bar\beta_2$ and $\bar\eta$.

Assuming incoherent excitation, it may be shown that

$$I_{\nu\nu}(0;\omega_0+\omega,\omega_1) = G_\nu M(\omega_1)R(\omega) \quad (19)$$

where $M(\omega_1)$ is the Fourier transform of the intensity modulation $m(t)$, $R_s(\omega-\omega_0)$ is the power spectrum of the unmodulated source, and G_ν is the initial mode power distribution. It is further assumed that the intensity modulation and the source spectrum are Gaussian-shaped

$$P_\nu(0;t) = G_\nu m(t) = G_\nu \exp[-(2t/\tau)^2]$$
$$R_s(\omega) = (2\pi)^{1/2}\exp[-(\omega^2/2B_s^2)]/B_s$$
$$r_s(t) = \boldsymbol{F}^{-1}\{R_s(\omega-\omega_0)\} = \exp[-(B_s t)^2/2] \quad (20)$$

corresponding to an incoherent source of relative band

$$\delta \equiv 2(2)^{1/2}B_s/\omega_0.$$

From (13) and (19)

$$c(\omega_0+\omega,\omega_1) = M(\omega_1)R(\omega)k_1 \quad (21)$$

where

$$k_1 \equiv \sum_\nu G_\nu B_\nu.$$

From (9), (13), (16), (17), and (21)

$$P_\nu(z;t) = k_1 B_\nu \exp(-\alpha_0 z)\cdot 2\,\mathrm{Re}\{W(z;t - z/v_0)\} \quad (22)$$

where

$$W(z;t) = \int_0^\infty \frac{d\omega}{2\pi}\exp(j\omega t)\{M(\omega)r_s(2\bar\beta_2\omega z)\cdot\exp[-\omega^2(\alpha_2 + j\bar\beta_2)z]\}.$$

It may be shown from (20) and (22) that

$$P_\nu(z;t) = \left(\frac{\tau}{T}\right)k_1 B_\nu \exp(-\alpha_0 z)\exp\left[-\left(\frac{t - z/v_0}{T/2}\right)^2\right] \quad (23)$$

where

$$T = (T_M^2 + \bar s^2 z^2)^{1/2}. \quad (24)$$

T_M accounts for multimode dispersion, and is given by [3]

$$T_M = (\tau^2 + 16\alpha_2 z)^{1/2}. \tag{25}$$

Equations (23)–(25) with $\bar{s} = 0$ yield the solution of [3]; $\bar{s}$ accounts for both the material and waveguide dispersion, and is given by

$$\bar{s} = \delta\bar{\eta}/c = (\delta/c) \sum_\nu B_\nu^2 \eta_\nu. \tag{26}$$

It is seen that each modal dispersion index η_ν is weighted by the square of the corresponding mode power in forming the average.

IV. PULSE SPREADING IN FIBERS WITH COUPLED DISPERSIVE MODES

Equations (24), (26), and (47) yield the following expression for the overall pulsewidth T:

$$T = [T_M^2 + (T_D - T_{\mathrm{WG}})^2]^{1/2} \tag{27}$$

where T_D is the dielectric dispersion

$$T_D = z\delta\eta_0/c \tag{28}$$

and

$$T_{\mathrm{WG}} = \frac{z\delta n_0 \Delta}{c}\left(\frac{\alpha}{\alpha+1}\right)\left(\frac{\alpha-2}{\alpha+2}\right). \tag{29}$$

Marcuse's pulsewidth T_M, modified by Personick [15] to include the effect of dielectric dispersion, yields

$$T_p = (T_M^2 + T_D^2)^{1/2}.$$

It follows that the percentage error incurred in neglecting the waveguide dispersion is given by

$$\text{error} = 100 \times \left(\frac{T - T_p}{T}\right). \tag{30}$$

It is now assumed that

$$T_M = RT_{\mathrm{MM}}, \qquad R < 1.0 \tag{31}$$

where R is the improvement factor [3] in pulsewidth due to mode coupling, and T_{MM} is the multimode dispersion in perfect fibers [11]

$$T_{\mathrm{MM}} = \frac{N_0 z \Delta}{c}\left(\frac{\alpha-2}{\alpha+2}\right), \qquad \alpha > 2 \tag{32}$$

calculated as the difference in the propagation times of the fastest and slowest modes.

The relative importance of waveguide dispersion for various fiber parameters may be inferred from Table I, which was calculated from (27)–(32) and (40), assuming also that $\delta = 0.04$, and $z = 1$ km.

TABLE I
ERROR INCURRED IN NEGLECTING WAVEGUIDE DISPERSION

(R,Δ)	α	2	3	5	10	∞
(0.5, 0.005)	% error	0	3	3	2	2
(0.1, 0.005)	% error	0	5	10	16	21
(0.1, 0.010)	% error	0	9	17	19	16

V. CONCLUSIONS

It has been found that taking waveguide dispersion into account may appreciably improve the fiber's calculated impulse response if the following two conditions are both satisfied.

1) The refractive-index distribution is such that multimode dispersion would predominate in the absence of mode coupling.

2) The mode coupling is sufficiently strong to reduce multimode dispersion to the level of dielectric dispersion.

APPENDIX
DISPERSION INDICES OF GRADIENT-INDEX FIBERS

For the class of index profiles given by [11]

$$n(r) = \begin{cases} n_0[1 - 2\Delta(r/a)^\alpha]^{1/2}, & r < a \\ n_0(1 - 2\Delta)^{1/2}, & r > a \end{cases} \tag{33}$$

it follows from [10] that:

$$n_\nu^0 \doteq n_0\{1 - \Delta(\nu/M)^{\alpha/(\alpha+2)}\}, \qquad \alpha > 2 \tag{34}$$

$$N_\nu^0 \doteq n_0\left\{1 + \Delta\left(\frac{\alpha-2}{\alpha+2}\right)(\nu/M)^{\alpha/(\alpha+2)}\right\}, \qquad \alpha > 2 \tag{35}$$

$$\eta_\nu^0 \doteq -\Delta n_0 \left(\frac{2\alpha}{\alpha+2}\right)\left(\frac{\alpha-2}{\alpha+2}\right)(\nu/M)^{\alpha/(\alpha+2)}, \qquad \alpha > 2 \tag{36}$$

where M is the total number of propagating modes, and the superscript 0 indicates that (34)–(36) are derived subject to the assumption that the bulk dielectric is nondispersive

$$\frac{dn(r)}{d\omega} = 0. \tag{37}$$

Assuming instead of (37) that [12]

$$\frac{d}{d\omega}[n(r)/n_0] = 0 \tag{38}$$

it is possible to write[1]

$$\left(\frac{\omega_0+\omega}{c}\right) n(r;\omega_0+\omega) \doteq \frac{\omega_0}{c}\left[n_0 + N_0\left(\frac{\omega}{\omega_0}\right) + \frac{\eta_0}{2}\left(\frac{\omega}{\omega_0}\right)^2\right] \times \begin{cases} [1 - 2\Delta(r/a)^\alpha]^{1/2}, & r < a \\ (1 - 2\Delta)^{1/2}, & r > a \end{cases} \tag{39}$$

analogous to (11). The three numbers (n_0, N_0, η_0) characterize the dispersive properties of the bulk dielectric; e.g., [9],

$$n_0 \doteq 1.45 \quad N_0 \doteq 1.47 \quad \eta_0 \doteq 0.025 \tag{40}$$

[1] It may be noted that (37) and (38) are equivalent only if $n_0 = N_0$ and $\eta_0 = 0$; otherwise (38) is more general.

for fused silica at $\lambda_0 = 0.8\ \mu$m.

Subject to (38) the results of [10] may be extended to show that

$$n_\nu = n_\nu^0 \tag{41}$$

$$N_\nu = (N_0/n_0)N_\nu^0 \tag{42}$$

$$\eta_\nu = \eta_\nu^0 - \left(\frac{N_0}{n_0} - 1\right)\left(\frac{N_\nu^0}{n_\nu} - 1\right)N_\nu^0 + \left(\frac{\eta_0}{N_0}\right)N_\nu^0. \tag{43}$$

From (43) and (34)–(36)

$$\eta_\nu \doteq \eta_0 - \Delta n_0 \left(\frac{2\alpha}{\alpha+2}\right)\left(\frac{\alpha-2}{\alpha+2}\right)(\nu/M)^{\alpha/(\alpha+2)}. \tag{44}$$

It may be noted that for step-index fibers ($\alpha = \infty$) (42) and (44) become

$$N_\nu = N_0\{1 + \Delta(\nu/M)\}$$

$$\eta_\nu = \eta_0 - 2\Delta n_0(\nu/M)$$

in agreement with [9, eqs. (15), (20)].

One may gain the impression that each of the M propagating modes has a different set of parameters (n_ν, N_ν, η_ν); however, this is not the case. As a result of mode degeneracies, ν is actually restricted to certain particular values, each of which corresponds to a group [13] of $W(\nu)$ modes with the same indices (n_ν, N_ν, η_ν). Utilizing the results of [14], it may be shown that the allowed values of ν are given by the squared integers: $4, 9, 16, \cdots, M$, and that

$$W(\nu) \doteq 2\nu^{1/2}. \tag{45}$$

Assuming that for sufficiently long fibers the power distributes itself equally among the modes

$$B_\nu^2 = 1/M. \tag{46}$$

From (26) and (44)–(46)

$$\bar{\eta} = \frac{1}{M}\sum_{\nu=4,9,\cdots}^{M} W(\nu)\eta_\nu \doteq \eta_0 - \Delta n_0\left(\frac{\alpha}{\alpha+1}\right)\left(\frac{\alpha-2}{\alpha+2}\right), \quad \alpha > 2. \tag{47}$$

Expressions analogous to (34)–(36) and (47) applicable to parabolic and near-parabolic profiles have not been given, as it may be shown that their waveguide dispersion is always negligible.

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REFERENCES

[1] S. E. Miller, E. A. J. Marcatili, and T. Li, "Research toward optical-fiber transmission systems—Part I: The transmission medium," *Proc. IEEE*, vol. 61, pp. 1703–1726, Dec. 1973.
[2] S. D. Personick, "Time dispersion in dielectric waveguides," *Bell Syst. Tech. J.*, vol. 50, pp. 843–859, Mar. 1971.
[3] D. Marcuse, *Theory of Dielectric Optical Waveguides*. New York: Academic, 1974.
[4] H. E. Rowe and D. T. Young, "Transmission distortion in multimode random waveguides," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-20, pp. 349–365, June 1972.
[5] J. A. Morrison and J. McKenna, "Coupled line equations with random coupling," *Bell Syst. Tech. J.*, vol. 51, pp. 209–228, Jan. 1972.
[6] S. D. Personick, "Two derivations of the time-dependent coupled-power equations," *Bell Syst. Tech. J.*, vol. 54, pp. 47–52, Jan. 1975.
[7] R. Steinberg, "Pulse propagation in multimode fibers with frequency-dependent coupling," *IEEE Trans. Microwave Theory Tech.* (*Special Issue on Integrated Optics and Optical Waveguides*), vol. MTT-23, pp. 121–122, Jan. 1975.
[8] D. T. Young and H. E. Rowe, "Optimum coupling for random guides with frequency-dependent coupling," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-20, pp. 365–372, June 1972.
[9] D. Gloge, "Dispersion in weakly guiding fibers," *Appl. Opt.*, vol. 10, pp. 2442–2445, Nov. 1971.
[10] F. P. Kapron, "Picosecond mode delays in gradient-refractive-index fibres," *Electron. Lett.*, vol. 10, pp. 235–237, June 1974. (Equations (5) and (8) in this paper are incomplete; cf.: R. A. Steinberg, "Comment on: Picosecond mode delays in gradient-refractive-index fibres," *Electron. Lett.*, vol. 10, pp. 375–376, Sept. 1974.)
[11] D. Gloge and E. A. J. Marcatili, "Multimode theory of graded-core fibers," *Bell Syst. Tech. J.*, vol. 52, pp. 1563–1578, Nov. 1973.
[12] K. Jurgensen, "Comment on: Dispersion minimization in dielectric waveguides," *Appl. Opt.*, vol. 13, pp. 1289–1290, June 1974.
[13] D. Gloge, "Optical power flow in multimode fibers," *Bell Syst. Tech. J.*, vol. 51, pp. 1767–1783, Oct. 1972.
[14] W. Streifer and C. N. Kurtz, "Scalar analysis of radially inhomogeneous guiding media," *J. Opt. Soc. Amer.*, vol. 57, pp. 779–786, June 1967.
[15] S. D. Personick, "Optimal trade-off of mode-mixing optical filtering and index difference in digital fiber optic communication systems," *Bell Syst. Tech. J.*, vol. 53, pp. 785–800, May–June 1974.